

EMBEDDING SUZUKI CURVES IN \mathbb{P}^4

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ABSTRACT. Here we study the projective geometry of smooth models $X_n \subseteq \mathbb{P}^4$ of plane Suzuki curves S_n . The knowledge of a system of generators for the Weierstrass semigroup at the only singular point of the curve is shown to have relevant geometric consequences. In particular, here we explicitly count the hypersurfaces of \mathbb{P}^4 containing X_n and provide a geometric characterization of those of small degree. We prove that the characterization cannot be extended to higher-degree hypersurfaces of \mathbb{P}^4 .

1. INTRODUCTION

Let $n \geq 2$ be an integer and let q_0 and q be defined by $q_0 := 2^n$, $q := 2q_0^2$. Let \mathbb{F}_q denote the finite field with q elements and fix any field \mathbb{F} containing \mathbb{F}_q . For the rest of the paper, \mathbb{F} will be the base field. Given an integer $r > 0$, we denote by \mathbb{P}^r the r -dimensional projective space over \mathbb{F} . The projective plane \mathbb{P}^2 will be referred to homogeneous coordinates $(x : y : z)$.

The **Suzuki curve** $S_n \subseteq \mathbb{P}^2$ associated to the integer n is defined over \mathbb{F} by the following affine equation:

$$y^q - y = x^{q_0}(x^q - x)$$

(see [7], Example 5.24). This curve is known to have only one point lying on the hyperplane at infinity $\{z = 0\}$, namely, $P_\infty := (0 : 1 : 0)$. This point, at which S_n has a cusp, is also the only singular point of the curve. The genus of S_n (i.e., by definition, the geometric genus of its normalization) is known to be $g_n := q_0(q - 1)$.

1.1. Main references on Suzuki curves. Suzuki curves are studied in depth throughout the book [7]. They are very interesting from a geometric viewpoint because of their optimality (Chapter 10) and their large group of automorphisms (Theorem 11.127 and, more generally, Section 12.2). Relevant properties of the Suzuki group date back to [5]. A comprehensive view on Suzuki curves and their quotients is given in [3]. On the same topics see also [8] and [10], Chapter V. More recently, the p -torsion group scheme of Jacobians of Suzuki curves has been studied in [1].

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Interesting applications of Suzuki curves in geometric Coding Theory have been successfully considered in [6] and in [9], computing also the Weierstrass semigroup associated to pairs of points of S_n ([9], Section III).

1.2. Layout of the paper. Here we consider a Suzuki curve S_n , as defined above, and its normalization $\pi : C_n \rightarrow S_n$. The normalization morphism, π , is known to be injective. In Section 2 we study linear systems of the form $|m\pi^{-1}(P_\infty)|$, $m \in \mathbb{Z}_{\geq 0}$. In particular, we give necessary and sufficient conditions for $|m\pi^{-1}(P_\infty)|$ to be very ample. The smallest integer m with this property is $q + 2q_0 + 1$. Moreover, the morphism induced by $|(q + 2q_0 + 1)\pi^{-1}(P_\infty)|$ embeds C_n into \mathbb{P}^4 . The curve obtained in this way, denoted by X_n , is a smooth model of S_n in \mathbb{P}^4 . The goal of the paper is to study the projective geometry of X_n . More precisely, we are interested in explicitly counting the hypersurfaces of \mathbb{P}^4 containing X_n and describing those of small degree. Our main result is the following one.

Theorem (see Theorem 27 and Corollary 28). Let X_n be the curve defined above and let $g_n = q_0(q - 1)$ be its genus. The following facts hold.

- (1) There exists a unique degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ containing X_n .
- (2) Let $2 \leq t \leq q_0$ be an integer. The degree t hypersurfaces of \mathbb{P}^4 containing X_n are exactly those containing Q_n . Moreover, they form an \mathbb{F} -vector space of dimension $\binom{t+4}{4} - \binom{t+2}{4}$.
- (3) The previous result is false for $t > q_0$. Indeed, there exist at least four linearly independent degree $q_0 + 1$ hypersurfaces of \mathbb{P}^4 containing X_n , and not containing Q_n .
- (4) Let $t \geq 2q_0 + 1$ be an integer. The degree t hypersurfaces of \mathbb{P}^4 containing X_n form an \mathbb{F} -vector space of dimension $\binom{t+4}{4} - t(q + 2q_0 + 1) - 1 + g_n$.

The theorem provides an interesting geometric characterization of the small-degree hypersurfaces of \mathbb{P}^4 containing X_n . Moreover it is proved that such a characterization cannot be extended to higher-degree hypersurfaces.

Remark 1. Two linearly independent hypersurfaces of \mathbb{P}^4 containing X_n and not containing Q_n appear in [1], page 4 (see also Example 29).

Section 3 and Section 4 are dedicated to preliminary results. In particular, in Section 3 we derive explicit formulas for the dimension of any Riemann-Roch space of the form $L(t(q + 2q_0 + 1))$, $t \in \mathbb{Z}_{\geq 0}$. On the other hand, in Section 4 we consider some multiplication maps of geometric interest, and study their properties. The computational results are interpreted from a geometric point of view in Section 5, leading to the main goal of the paper.

Remark 2. The linear series $|(q + 2q_0 + 1)\pi^{-1}(P_\infty)|$ here considered is of deep interest in the literature. Its properties can be used to characterize Suzuki curves in terms of the genus and the number of rational points (see [7], Theorem 10.102).

2. GEOMETRY ON THE WEIERSTRASS SEMIGROUP

Given a Suzuki curve S_n and an integer $m \geq 0$, we denote by $L(mP_\infty)$ the vector space of the rational functions on S_n whose pole order at P_∞ is at most m , i.e., the Riemann-Roch space associated to the divisor mP_∞ on S_n . We recall that the Weierstrass semigroup $H(P_\infty)$ associated to P_∞ is precisely the set of non-gaps at P_∞ . In other words, $H(P_\infty)$ is the set of all the $m \in \mathbb{Z}_{\geq 0}$ such that there exists a rational function in $L(mP_\infty) \setminus L((m - 1)P_\infty)$.

Remark 3. Since, for any $m \geq 0$, we have $0 \leq L((m + 1)P_\infty) - L(mP_\infty) \leq 1$, by definition of Weierstrass semigroup we clearly get $\dim_{\mathbb{F}} L(mP_\infty) = |\{s \in H(P_\infty) : s \leq m\}|$.

Lemma 4 ([9], Lemma 3.1). Let $H(P_\infty)$ be the Weierstrass semigroup defined above. Then $H(P_\infty) = \langle q, q + q_0, q + 2q_0, q + 2q_0 + 1 \rangle$.

Notation 5. For any $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$, we set

$$\|(a, b, c, d)\| := aq + b(q + q_0) + c(q + 2q_0) + d(q + 2q_0 + 1).$$

Notation 6. The normalization of S_n will be always denoted by C_n . It is well-known (see for instance [2], Section 7.5) that C_n is a smooth abstract curve which is birational to S_n . The normalization morphism $\pi : C_n \rightarrow S_n$ is here injective. Hence we will simply write P_∞ instead of $\pi^{-1}(P_\infty)$.

Here we focus on C_n curves and study linear systems on them of the form $|mP_\infty|$, providing a characterization of the very ample ones.

Lemma 7. Let m be a positive integer. The linear system $|mP_\infty|$ is spanned by its global sections if and only if $m \in H(P_\infty)$.

Proof. This is a well-known property of the one-point Weierstrass semigroup $H(P_\infty)$ (notice that C_n is smooth). \square

Proposition 8. Let m be a positive integer. The linear system $|mP_\infty|$ is very ample if and only if $m \in H(P_\infty)$ and $m - 1 \in H(P_\infty)$.

Proof. If $|mP_\infty|$ is very ample, then it is obviously spanned by its global sections. Hence, by Lemma 7, we get $m \in H(P_\infty)$. Let $r := \dim_{\mathbb{F}} L(mP_\infty)$ and denote by $\varphi_m : C_n \rightarrow \mathbb{P}^{r-1}$ the morphism induced by mP_∞ . The linear system $|mP_\infty|$ is very ample if and only if φ_m is injective with non-zero differential at any point of C_n .

(\Rightarrow) Assume that the linear system $|mP_\infty|$ is very ample. In particular, φ_m must have non-zero differential at P_∞ . This implies the existence of a rational function $f \in L(mP_\infty)$ whose vanishing order at P_∞ is exactly one. Since $m \in H(P_\infty)$, this implies $m - 1 \in H(P_\infty)$.

(\Leftarrow) On the other hand, assume $m, m - 1 \in H(P_\infty)$. We clearly have $m \geq q + 2q_0 + 1$. As in Notation 6, let $\pi : C_n \rightarrow S_n$ denote the normalization morphism of S_n . Since $(x)_\infty = qP_\infty$ and $(y)_\infty = (q + q_0)P_\infty$ (see [6], Proposition 1.3), we have $\{1, x, y\} \subseteq L(mP_\infty)$. Hence the linear system $|mP_\infty|$ contains the linear system spanned by $\{1, x, y\}$, which induces the composition of π with the inclusion $S_n \hookrightarrow \mathbb{P}^2$. Since P_∞ is the only singular point of S_n , the morphism φ_m is injective with non-zero differential at any point of $C_n \setminus \{P_\infty\}$. Therefore, in order to prove that $|mP_\infty|$ is very ample, it is necessary and sufficient to show that $\dim_{\mathbb{F}} L((m - 2)P_\infty) = \dim_{\mathbb{F}} L(mP_\infty) - 2$. Since $m, m - 1 \in H(P_\infty)$, this condition is clearly satisfied. \square

Remark 9. Proposition 8 shows that the smallest projective space in which C_n can be embedded by a one-point linear system $|mP_\infty|$ is \mathbb{P}^4 .

3. RIEMANN-ROCH SPACES OF SUZUKI CURVES

In this section we provide an explicit formula for the dimension of any Riemann-Roch space of the form $L(t(q + 2q_0 + 1)P_\infty)$, $t \in \mathbb{Z}_{\geq 0}$. Since the Weierstrass semigroup $H(P_\infty)$ is known (Proposition 4), the dimension of $L(mP_\infty)$ is also known, *in principle*, for any $m \geq 0$. On the other hand, deriving easy-handable expressions from the semigroup's data is not completely

trivial. Explicit formulas and their combination are key-points in the proofs of this paper. The main results of the Section are Proposition 14 and Proposition 18, whose proofs are splitted in some preliminary lemmas.

Lemma 10. Let $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ and let $t \leq q_0 - 1$ be a positive integer. The following two facts are equivalent:

- (A) $\|(a, b, c, d)\| \leq t(q + 2q_0 + 1)$,
- (B) $a + b + c + d \leq t$.

Proof. Assume $a + b + c + d \leq t$. Then $\|(a, b, c, d)\| \leq (a + b + c + d)(q + 2q_0 + 1) \leq t(q + 2q_0 + 1)$. On the other hand, we may note that $q > q - q_0 - 1 = 2(q_0 - 1)q_0 + q_0 - 1 \geq 2tq_0 + t$. Hence $(t + 1)q > t(q + 2q_0 + 1)$. If $a + b + c + d > t$ then $a + b + c + d \geq t + 1$. As a consequence, we get $\|(a, b, c, d)\| \geq (a + b + c + d)q \geq (t + 1)q > t(q + 2q_0 + 1)$. \square

Lemma 11. Let $(a', b', c', d') \in \mathbb{Z}_{\geq 0}^4$. Choose any integer t with $1 \leq t \leq q_0 - 1$ and assume $\|(a', b', c', d')\| \leq t(q + 2q_0 + 1)$. There exists a unique 4-tuple $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ with $b \in \{0, 1\}$ and $\|(a, b, c, d)\| = \|(a', b', c', d')\|$.

Proof. First of all, we prove the existence. Write $b' = 2\beta + B$, with $\beta \geq 0$ and $r \in \{0, 1\}$, and set $a := a' + \beta$, $b := B$, $c := c' + \beta$, $d := d'$. Now we prove the uniqueness. Assume that there exist $(a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2) \in \mathbb{Z}_{\geq 0}^4$ such that:

- (A) $b_1, b_2 \in \{0, 1\}$,
- (B) $\|(a_1, b_1, c_1, d_1)\| = \|(a_2, b_2, c_2, d_2)\|$,
- (C) $\|(a_1, b_1, c_1, d_1)\|, \|(a_2, b_2, c_2, d_2)\| \leq t(q + 2q_0 + 1)$.

As in the proof of Lemma 10, we have $(t + 1)q > t(q + 2q_0 + 1)$. Condition (iii) implies, in particular, $c_1, c_2, d_1, d_2 \leq t \leq q_0 - 1$. Condition (ii) is equivalent to

$$(1) \quad (a_1 - a_2)q + (b_1 - b_2)(q + q_0) + (c_1 - c_2)(q + 2q_0) + (d_1 - d_2)(q + 2q_0 + 1) = 0.$$

Reducing modulo q_0 , we have $d_1 - d_2 \equiv 0 \pmod{q_0}$. Since $-q_0 + 1 \leq d_1, d_2 \leq q_0 - 1$ we deduce $d_1 = d_2$. Hence equation (1) becomes

$$(2) \quad (a_1 - a_2)q + (b_1 - b_2)(q + q_0) + (c_1 - c_2)(q + 2q_0) = 0.$$

Reducing modulo $2q_0$, we obtain $(b_1 - b_2)q_0 \equiv 0 \pmod{2q_0}$. Since $b_1, b_2 \in \{0, 1\}$, one gets $b_1 = b_2$. By substitution into equation (2), we may write

$$(3) \quad (a_1 - a_2)q + (c_1 - c_2)(q + 2q_0) = 0.$$

Reducing modulo q , we get $(c_1 - c_2)2q_0 \equiv 0 \pmod{q}$. Since $q = 2q_0^2$ and $c_1, c_2 \leq q_0 - 1$, we conclude $c_1 = c_2$. Clearly $a_1 = a_2$ at this point. \square

Remark 12. The uniqueness argument in the proof of the previous lemma is applied also in [1], Proposition 3.7, to get an analogous result.

The following lemma summarizes some trivial enumeration facts we are going to apply. A proof can be easily obtained by induction.

Lemma 13. Let h be a positive integer. The following formulas hold.

- (A) $\sum_{i=0}^h i = h(h + 1)/2$.
- (B) $\sum_{i=0}^h i^2 = h^3/3 + h^2/2 + h/6$.
- (C) Let \mathcal{T}_h be the set of all the 3-tuple $(a, b, c) \in \mathbb{Z}_{\geq 0}^3$ satisfying $a + b + c = h$. We have $|\mathcal{T}_h| = (h + 1)(h + 2)/2$.

Proposition 14. Let t be a non-negative integer and let $g_n = q_0(q - 1)$ be the genus of the Suzuki curve S_n (see the Introduction). The dimension of the one-point Riemann-Roch space $L(t(q + 2q_0 + 1)P_\infty)$ is given by the following formulas:

$$\dim_{\mathbb{F}} L(t(q + 2q_0 + 1)P_\infty) = \begin{cases} 4t + 1 & \text{if } t = 0 \text{ or } t = 1, \\ \binom{t+4}{4} - \binom{t+2}{4} & \text{if } 2 \leq t \leq q_0 - 1, \\ t(q + 2q_0 + 1) + 1 - g_n + \binom{2q_0 - t + 2}{4} - \binom{2q_0 - t}{4} & \text{if } q_0 \leq t \leq 2q_0 - 4, \\ t(q + 2q_0 + 1) + 6 - g_n & \text{if } t = 2q_0 - 3, \\ t(q + 2q_0 + 1) + 2 - g_n & \text{if } t = 2q_0 - 2, \\ t(q + 2q_0 + 1) + 1 - g_n & \text{if } t \geq 2q_0 - 1. \end{cases}$$

Proof. We recall (Remark 3) that $\dim_{\mathbb{F}} L(t(q + 2q_0 + 1))$ is exactly the cardinality of the set $H_t(P_\infty) := \{s \in H(P_\infty) : s \leq t(q + 2q_0 + 1)\}$. The proof is divided into five steps.

(A) If $t = 0, 1$ the dimension is easily computed by hands (Lemma 4).

(B) Assume $2 \leq t \leq q_0 - 1$. Combining Lemma 10 and Lemma 11 we see that, for any $t \in \{2, \dots, q_0 - 1\}$, the cardinality of $H_t(P_\infty)$ may be computed as

$$|H_t(P_\infty)| = |\{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 : b \in \{0, 1\} \text{ and } a + b + c + d \leq t\}|.$$

Hence, following the notation of Lemma 13, we write

$$\begin{aligned} |H_t(P_\infty)| &= \sum_{h=0}^t |\{(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4 : b \in \{0, 1\} \text{ and } a + b + c + d = h\}| \\ &= \sum_{h=0}^t |\mathcal{T}_h| + \sum_{h=1}^t |\mathcal{T}_{h-1}| \\ &= |\mathcal{T}_t| + 2 \sum_{h=0}^{t-1} |\mathcal{T}_h| \\ &= (t+1)(t+2)/2 + \sum_{h=0}^{t-1} h^2 + 3h + 2 \\ &= (2t^3 + 9t^2 + 13t + 6)/6 \\ &= \binom{t+4}{4} - \binom{t+2}{4}, \end{aligned}$$

which is the expected formula.

(C) Since the genus of S_n is $g_n = q_0(q - 1)$, we compute $2g_n - 2 = 2(q_0 - 1)(q + 2q_0 + 1)$. Hence, for $t \geq 2q_0 - 2$, the dimension of $L(t(q + 2q_0 + 1))$ is given by a trivial application of the Riemann-Roch Theorem and the fact that $\dim_{\mathbb{F}} L(0) = 1$.

(D) Here we assume $q_0 \leq t \leq 2q_0 - 4$ and set $D_t := t(q + 2q_0 + 1)P_\infty$. A canonical divisor on S_n is $K = (2g_n - 2)P_\infty \sim 2(q_0 - 1)(q + 2q_0 + 1)P_\infty$. See also [1] for details. We have a linear equivalence of divisors

$$K - D_t \sim (2q_0 - 2 - t)(q + 2q_0 + 1)P_\infty.$$

Since $2 \leq 2q_0 - 2 - t \leq q_0 - 1$, thanks to step (B) we are able to explicitly compute $\dim_{\mathbb{F}} L(K - D_t)$ and obtain $\dim_{\mathbb{F}} L(D_t)$ by applying the Riemann-Roch Theorem as follows:

$$\dim_{\mathbb{F}} L(D_t) = t(q + 2q_0 + 1) + 1 - g_n + \binom{2q_0 - t + 2}{4} - \binom{2q_0 - t}{4}.$$

(E) Finally, assume $t = 2q_0 - 3$ and set $D := (2q_0 - 3)(q + 2q_0 + 1)P_\infty$. We have a linear equivalence $K - D \sim (q + 2q_0 + 1)P_\infty$ and so, by step (A), the dimension of $L(D)$ is again computed by the Riemann-Roch Theorem. \square

We conclude this section providing an explicit monomial basis of any Riemann-Roch space $L(mP_\infty)$, $m \geq 0$. The following preliminary result generalizes Lemma 11.

Lemma 15. Let $(a', b', c', d') \in \mathbb{Z}_{\geq 0}^4$. There exists a unique $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ which satisfies the following properties:

$$0 \leq b \leq 1, \quad 0 \leq c \leq q_0 - 1, \quad 0 \leq d \leq q_0 - 1, \quad \|(a, b, c, d)\| = \|(a', b', c', d')\|.$$

Proof. To prove the uniqueness we may apply the same argument as Lemma 11, which uses only our hypothesis on b , c and d . Let us prove the existence. Write $d' = \delta q_0 + D$ with $0 \leq D \leq q_0 - 1$ and set $(a_1, b_1, c_1, d_1) := (a' + \delta q_0, b' + \delta, c', D)$. Write $b_1 = 2\beta + B$ with $0 \leq B \leq 1$, and set $(a_2, b_2, c_2, d_2) := (a_1 + \beta, B, c_1 + \beta, D)$. Write $c_2 = \gamma q_0 + C$ with $0 \leq C \leq q_0 - 1$, and define

$$(a, b, c, d) := (a_2 + \gamma q_0 + \gamma, b_2, C, d_2) = (a' + \delta q_0 + \gamma q_0 + \beta + \gamma, B, C, D).$$

It is easily checked that (a, b, c, d) has the expected properties. \square

Definition 16. Following [6] and [9], we define the rational functions $v := y^{2q_0} + x^{2q_0+1}$ and $w := y^{2q_0}x + v^{2q_0}$. The pole divisors of x, y, v, w are computed in [6], Proposition 1.3:

$$(x)_\infty = qP_\infty, \quad (y)_\infty = (q + q_0)P_\infty, \quad (v)_\infty = (q + 2q_0)P_\infty, \quad (w)_\infty = (q + 2q_0 + 1)P_\infty.$$

Remark 17. From the pole divisors given in the previous definition we see that, for any $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$, the pole order of $x^a y^b v^c w^d$ at P_∞ is exactly $\|(a, b, c, d)\|$.

Proposition 18. Let $m \geq 0$ be an integer. A basis of the Riemann-Roch space $L(mP_\infty)$ is given by all the rational functions $x^a y^b v^c w^d$ such that:

$$a, b, c, d \in \mathbb{Z}_{\geq 0}, \quad 0 \leq b \leq 1, \quad 0 \leq c, d \leq q_0 - 1, \quad \|(a, b, c, d)\| \leq m.$$

Proof. By Lemma 15, such rational functions have different pole orders at P_∞ . In particular, they are linearly independent. By Remark 17, they all belong to $L(mP_\infty)$. Finally, by definition of $H(P_\infty)$ and Lemma 15, their number is $\dim_{\mathbb{F}} L(mP_\infty)$. Hence they form a basis of the Riemann-Roch space $L(mP_\infty)$. \square

4. MULTIPLICATION MAPS AND THEIR GEOMETRY

Let S_n be the Suzuki curve defined in the Introduction and let $\pi : C_n \rightarrow S_n$ be its normalization (see Notation 6). By Proposition 8, the linear system $|(q + 2q_0 + 1)P_\infty|$ defines an embedding $\varphi_{q+2q_0+1} : C_n \rightarrow \mathbb{P}^4$. We set $X_n := \varphi_{q+2q_0+1}(C_n)$, a smooth curve of degree $q + 2q_0 + 1$ in \mathbb{P}^4 .

Definition 19. Given non-negative integers a , b and t , we will denote by $\mu(a, b)$ and $\mu_t(a)$, respectively, the multiplication maps

$$\mu(a, b) : L(aP_\infty) \otimes L(bP_\infty) \rightarrow L((a + b)P_\infty), \quad \mu_t(a) : L(aP_\infty)^{\otimes t} \rightarrow L(taP_\infty).$$

Since in the function field defined by S_n multiplication is commutative, each of the maps $\mu_t(a)$ induces a multiplication map $\sigma_t(a) : S^t(L(aP_\infty)) \rightarrow L(taP_\infty)$, where $S^t(L(aP_\infty))$ denotes the t -th power of the symmetric tensor product.

Remark 20. This section is rather technical. Here we study the surjectivity of the multiplication maps $\sigma_t(q+2q_0+1)$, $t \geq 1$, introduced in Definition 19. Interesting geometric applications will be shown later in the paper. The main results of this section are Proposition 25 and its consequences (Corollary 26). The proof of the cited proposition is splitted in Lemmas 21, 22, 23 and 24.

Lemma 21. Let α and β be non negative integers such that $\alpha + \beta \leq q_0 - 1$. The multiplication map $\mu(\alpha(q+2q_0+1), \beta(q+2q_0+1))$ of Definition 19 is surjective.

Proof. Since α and β play interchangeable roles and the case $\alpha = 0$ is trivial, we may assume $\beta \geq \alpha > 0$. Keep on mind Proposition 18 and consider a basis element, $x^a y^b v^c w^d$, of the Riemann-Roch space $L((\alpha + \beta)(q+2q_0+1)P_\infty)$. We clearly have

$$(4) \quad aq + b(q+q_0) + c(q+2q_0) + d(q+2q_0+1) \leq (\alpha + \beta)(q+2q_0+1).$$

Since $\alpha + \beta \leq q_0 - 1$, we get $\alpha + \beta \leq q_0 - 1 < 2q_0^2/(2q_0+1) = q/(2q_0+1)$. As a consequence, $(\alpha + \beta)(2q_0+1) < q$, i.e., $q(\alpha + \beta + 1) > (\alpha + \beta)(q+2q_0+1)$. By inequality (4) we have, in particular, $(a+b+c+d)q \leq (\alpha + \beta)(q+2q_0+1) < q(\alpha + \beta + 1)$. Dividing by q one obtains $a+b+c+d < \alpha + \beta + 1$ and so $a+b+c+d \leq \alpha + \beta$. Now we write $(a, b, c, d) = (a_1, b_1, c_1, d_1) + (a_2, b_2, c_2, d_2)$ with $a_1 + b_1 + c_1 + d_1 \leq \alpha$ and $a_2 + b_2 + c_2 + d_2 \leq \beta$. It follows $\|(a_1, b_1, c_1, d_1)\| \leq \alpha(q+2q_0+1)$, $\|(a_2, b_2, c_2, d_2)\| \leq \beta(q+2q_0+1)$ and so

$$x^a y^b v^c w^d = \mu(\alpha(q+2q_0+1), \beta(q+2q_0+1)) (x^{a_1} y^{b_1} v^{c_1} w^{d_1} \otimes x^{a_2} y^{b_2} v^{c_2} w^{d_2}).$$

In other words, a generic basis element $x^a y^b v^c w^d \in L((\alpha + \beta)(q+2q_0+1)P_\infty)$ is in the image of $\mu(\alpha(q+2q_0+1), \beta(q+2q_0+1))$, as claimed. \square

Lemma 22. Let $t \geq 1$ be an integer. We follow the notation of Lemma 13. Let $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ with $a + b + c + d \leq t$. There exist 4-tuple $\{(a_i, b_i, c_i, d_i)\}_{i=1}^t \subseteq \mathcal{T}_1$ such that

$$(a, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i).$$

Proof. We use induction on t . If $t = 1$ then we take $(a_1, b_1, c_1, d_1) := (a, b, c, d)$. Now assume $a + b + c + d \leq t + 1$. If $a + b + c + d \leq t$ then, by inductive hypothesis, we write $(a, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$ with $\{(a_i, b_i, c_i, d_i)\}_{i=1}^t \subseteq \mathcal{T}_1$. As a consequence, we may define the 4-tuple $(a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1}) := (0, 0, 0, 0) \in \mathcal{T}_1$ and obtain $(a, b, c, d) = \sum_{i=1}^{t+1} (a_i, b_i, c_i, d_i)$. On the other hand, if $a + b + c + d = t + 1$ then one among a, b, c, d must be positive. Assume without restriction $a > 0$. Then, by induction, $(a-1, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$ with $\{(a_i, b_i, c_i, d_i)\}_{i=1}^t \subseteq \mathcal{T}_1$. By setting $(a_{t+1}, b_{t+1}, c_{t+1}, d_{t+1}) := (1, 0, 0, 0) \in \mathcal{T}_1$, we have $(a, b, c, d) = \sum_{i=1}^{t+1} (a_i, b_i, c_i, d_i)$, and the lemma is proved. \square

Lemma 23. Let m be a positive integer. Let $\{f_1, \dots, f_h\} \subseteq L(mP_\infty)$ be a set of rational functions such that for any $s \in H(P_\infty)$, with $s \leq m$, there exists a $1 \leq j \leq h$ such that $(f_j)_\infty = s$. Then $\{f_1, \dots, f_h\}$ is a generating set of $L(mP_\infty)$.

Proof. For any $s \in H(P)$, with $s \leq m$, there exists a $1 \leq j_s \leq h$ such that $(f_{j_s})_\infty = sP_\infty$. Hence $\mathcal{B}_m := \{f_{j_s} : s \in H(P_\infty), s \leq m\}$ is a set of linearly independent elements of $L(mP_\infty)$ whose cardinality is $\dim_{\mathbb{F}} L(mP_\infty)$. Indeed, its elements are rational functions whose evaluations at P_∞ are distinct. Hence \mathcal{B}_m is a basis of $L(mP_\infty)$. We conclude by observing that $\{f_1, \dots, f_h\}$ contains \mathcal{B}_m . \square

Lemma 24. Let $t \geq 2q_0 + 1$ be an integer. For any $s \in H(P_\infty)$, with $s \leq t(q+2q_0+1)$, there exists a 4-tuple $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ such that $\|(a, b, c, d)\| = s$ and $a + b + c + d \leq t$.

Proof. The argument is divided into two steps.

- (A) Here we assume $s \leq tq$ and take any 4-tuple (a, b, c, d) such that $\|(a, b, c, d)\| = s$; such a 4-tuple exists because $s \in H(P_\infty)$. If $a + b + c + d \geq t + 1$, then we clearly have the contradiction $s = \|(a, b, c, d)\| \geq (t + 1)q > s$. Hence $a + b + c + d \leq t$, and we are done.
- (B) Now assume $s > tq$. Write $s = \alpha(q + 2q_0 + 1) - \beta$ with $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $0 \leq \beta \leq q + 2q_0$. Since $s \leq t(q + 2q_0 + 1)$, we have $0 \leq \alpha \leq t$. Set:

$$e_1 := \left\lfloor \frac{\beta}{2q_0 + 1} \right\rfloor, \quad e_2 := \left\lfloor \frac{\beta - e_1(2q_0 + 1)}{q_0 + 1} \right\rfloor, \quad e_3 := \beta - e_1(2q_0 + 1) - e_2(q_0 + 1).$$

Notice that $e_2 \in \{0, 1\}$ and $e_3 \leq q_0$. Since $\beta \leq q + 2q_0 = (2q_0 + 1)q_0$, we get $e_1 \leq q_0$ and the equality holds if and only if $\beta = q + 2q_0$. In this case we have $e_2 = \lfloor q_0/(q_0 + 1) \rfloor = 0$. Hence, in any case, $e_1 + e_2 + e_3 \leq 2q_0$. We may also observe

$$s > tq \geq (2q_0 + 1)q > (2q_0 - 1)(q + 2q_0 + 1).$$

Since $s = \alpha(q + 2q_0 + 1) - \beta$ with $0 \leq \beta \leq q + 2q_0$, we deduce $\alpha \geq 2q_0 \geq e_1 + e_2 + e_3$. Hence $e_1 + e_2 + e_3 \leq \alpha \leq t$ and we may take $a := e_1$, $b := e_2$, $c := e_3$ and $d := \alpha - e_1 - e_2 - e_3$ to conclude the proof. \square

Proposition 25. Let t be a positive integer and let $\sigma_t(q + 2q_0 + 1)$ be as in Definition 19.

- (1) If $1 \leq t \leq q_0$ then $\sigma_t(q + 2q_0 + 1)$ is surjective.
- (2) If $t \geq 2q_0 + 1$ then $\sigma_t(q + 2q_0 + 1)$ is surjective.

Proof. Let us divide the proof into three steps.

- (A) Here we assume $1 \leq t \leq q_0 - 1$. In the notations of Definition 19, the image of the map $\sigma_t(q + 2q_0 + 1)$ and the image of the map $\mu_t(q + 2q_0 + 1)$ coincide. Moreover, the image of $\mu_t(q + 2q_0 + 1)$ contains the image of $\mu(q + 2q_0 + 1, (t - 1)(q + 2q_0 + 1))$. Since $t = 1 + (t - 1) \leq q_0 - 1$, by Lemma 21 the map $\mu(q + 2q_0 + 1, (t - 1)(q + 2q_0 + 1))$ is surjective, and we are done.
- (B) Assume $t \geq 2q_0 + 1$. We recall that $L(q + 2q_0 + 1)$ has $\{1, x, y, v, w\}$ as a basis. Moreover, $1, x, y, v, w$ have the following pole divisors:

$$(1)_\infty = 0, \quad (x)_\infty = q, \quad (y)_\infty = q + q_0, \quad (v)_\infty = q + 2q_0, \quad (w)_\infty = q + 2q_0 + 1.$$

Take any $s \in H(P_\infty)$ with $s \leq t(q + 2q_0 + 1)$. By Lemma 24 there exists a 4-tuple $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$ such that $\|(a, b, c, d)\| = s$ and $a + b + c + d \leq t$. Thanks to Lemma 22, we write $(a, b, c, d) = \sum_{i=1}^t (a_i, b_i, c_i, d_i)$, with $(a_i, b_i, c_i, d_i) \in \mathcal{T}_1$ for any $i \in \{1, \dots, t\}$. Hence, for any $i \in \{1, \dots, t\}$, we have $x^{a_i} y^{b_i} v^{c_i} w^{d_i} \in L((q + 2q_0 + 1)P_\infty)$. Moreover,

$$\sigma_t(q + 2q_0 + 1) \left(\bigotimes_{i=1}^t x^{a_i} y^{b_i} v^{c_i} w^{d_i} \right) = x^a y^b v^c w^d$$

is a rational function in the image of σ_t whose pole divisor is exactly sP_∞ . Notice that s is arbitrary in $H(P_\infty)$ with $s \leq t(q + 2q_0 + 1)$. Hence, by Lemma 23, the image of $\sigma_t(q + 2q_0 + 1)$ spans the vector space $L(t(q + 2q_0 + 1)P_\infty)$, i.e., $\sigma_t(q + 2q_0 + 1)$ is surjective.

- (C) Let us consider the case $t = q_0$. By Lemma 23, it is enough to prove that for any $s \in H(P_\infty)$, with $s \leq q_0(q + 2q_0 + 1)$, there exists a rational function, say f , in the image of $\sigma_{q_0}(q + 2q_0 + 1)$ with the property $(f)_\infty = sP_\infty$. Write $s = \|(a, b, c, d)\|$ for a certain $(a, b, c, d) \in \mathbb{Z}_{\geq 0}^4$.

- (C.1) If $s \leq (q_0 - 1)(q + 2q_0 + 1)$ then, by Lemma 10, we conclude $a + b + c + d \leq q_0 - 1 < q_0$.

In this case we are done, as in step (B).

(C.2) Assume $s > (q_0 - 1)(q + 2q_0 + 1)$. If $a + b + c + d \leq q_0$ then, as above, the result is proved. Hence we will assume $a + b + c + d \geq q_0 + 1$ for the rest of the proof. If $a + b + c + d \geq q_0 + 2$ then $\|(a, b, c, d)\| \geq (q_0 + 2)q > q_0(q + 2q_0 + 1)$. As a consequence, we have $a + b + c + d \leq q_0 + 1$ and so $a + b + c + d = q_0 + 1$. Assume $a \leq q_0 - 1$. Then $b + c + d \geq 2$ and so $\|(a, b, c, d)\| \geq (q_0 + 1)q + 2q_0 > q_0(q + 2q_0 + 1)$, a contradiction. It follows $a \in \{q_0, q_0 + 1\}$, and we can study the two cases separately.

- If $a = q_0 + 1$ then clearly $b = c = d = 0$ and so x^{q_0+1} is a rational function with the expected pole divisor. Working modulo the equation of S_n , we have $x^{q_0+1} = v^{q_0} - y$ (see [1], page 4). Since v^{q_0} and y trivially belong to the image of $\sigma_{q_0}(q + 2q_0 + 1)$, x^{q_0+1} also belongs to such image.
- If $a = q_0$ and $(c, d) \neq (0, 0)$, then $\|(a, b, c, d)\| \geq q_0q + (q + 2q_0) > q_0(q + 2q_0 + 1)$, a contradiction. It follows $c = d = 0$ and $b = 1$. Notice that $x^{q_0}y$ is a rational function with the expected pole divisor. Moreover, $x^qy = w^{q_0} - v$ (again [1], page 4) and so we conclude as in the previous step. \square

Corollary 26. Let t be an integer. If $1 \leq t \leq q_0$ or $t \geq 2q_0 + 1$ then the restriction map of cohomology groups $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow H^0(X_n, \mathcal{O}_{X_n}(t))$ is surjective.

Proof. Since the embedding of curves $\varphi_{q+2q_0+1} : C_n \rightarrow X_n$ is induced by the linear system $|(q + 2q_0 + 1)P_\infty|$, the pull-back bundle of $\mathcal{O}_{X_n}(1)$ through φ_{q+2q_0+1} is that associated to the linear system $|(q + 2q_0 + 1)P_\infty|$. By Proposition 25, the restriction map $S^t(H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))) \rightarrow H^0(X_n, \mathcal{O}_{X_n}(t))$ is surjective. The thesis follows. \square

5. THE SMOOTH MODEL OF A SUZUKI CURVE IN \mathbb{P}^4

Here we study the geometric properties of the smooth model $X_n \subseteq \mathbb{P}^4$ of a Suzuki curve S_n . We apply the computational results derived in the previous parts of the paper in order to count the hypersurfaces of \mathbb{P}^4 containing X_n (Theorem 27). Moreover, we provide an explicit geometric characterization of those of small degree (Corollary 28).

Theorem 27. Let t be a positive integer and let $\mathcal{K}(t, X_n)$ denote the \mathbb{F} -vector space of all the degree t hypersurfaces of \mathbb{P}^4 containing X_n . Let $\kappa(t, X_n)$ be the dimension of $\mathcal{K}(t, X_n)$. The following formulas hold:

$$\kappa(t, X_n) = \begin{cases} \binom{t+4}{4} - \binom{t+2}{4} & \text{if } 2 \leq t \leq q_0, \\ \binom{t+4}{4} - t(q + 2q_0 + 1) - 1 + g_n & \text{if } t \geq 2q_0 + 1. \end{cases}$$

Proof. The vector space $\mathcal{K}(t, X_n)$, whose dimension is in question, is exactly the kernel of the restriction map $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow H^0(X_n, \mathcal{O}_{X_n}(t))$. If $2 \leq t \leq q_0$ or $t \geq 2q_0 + 1$, by Corollary 26, ρ_t is surjective. It follows

$$\kappa(t, X_n) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) - h^0(X_n, \mathcal{O}_{X_n}(t)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) - \dim_{\mathbb{F}} L(t(q + 2q_0 + 1)P_\infty).$$

Now use the formulas given in Proposition 14. \square

Theorem 27 allows us to geometrically characterize all the small-degree hypersurfaces of \mathbb{P}^4 containing X_n .

Corollary 28. Let X_n be the smooth projective model of the Suzuki curve S_n in \mathbb{P}^4 , obtained through the linear system $|(q + 2q_0 + 1)P_\infty|$. The following facts hold.

- (1) There exists a unique degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ containing X_n .
- (2) Let $2 \leq t \leq q_0$ be an integer. The degree t hypersurfaces of \mathbb{P}^4 containing X_n are exactly those containing Q_n . Moreover, they form an \mathbb{F} -vector space of dimension $\binom{t+4}{4} - \binom{t+2}{4}$.

- (3) There exist at least four linearly independent degree $q_0 + 1$ hypersurfaces of \mathbb{P}^4 containing X_n and not containing Q_n .

Proof. Let $t \geq 2$ be any integer. Since $h^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-2)) = 0$, the well-known exact sequence of sheaves $0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(t-2) \rightarrow \mathcal{O}_{\mathbb{P}^4}(t) \rightarrow \mathcal{O}_{Q_n}(t) \rightarrow 0$ gives that the induced restriction map of cohomology groups $\rho'_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow H^0(Q_n, \mathcal{O}_{Q_n}(t))$ is surjective, and allows us to compute $h^0(Q_n, \mathcal{O}_{Q_n}(t)) = \binom{t+4}{4} - \binom{t+2}{4}$. The restriction map $\rho_t : H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow H^0(X_n, \mathcal{O}_{X_n}(t))$ factors through ρ'_t . More precisely, we have a commutative diagram of restriction maps between cohomology groups:

$$\begin{array}{ccc} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) & \xrightarrow{\rho'_t} & H^0(Q_n, \mathcal{O}_{Q_n}(t)) \\ & \searrow \rho_t & \downarrow \rho''_t \\ & & H^0(X_n, \mathcal{O}_{X_n}(t)) \end{array}$$

(we recall that Q_n contains X_n). Since ρ'_t is surjective, we clearly have $\text{Im}(\rho_t) = \text{Im}(\rho''_t)$. Let us divide the rest of the proof into two steps.

- (A) Assume $2 \leq t \leq q_0$. We proved in Corollary 26 that the restriction map ρ_t is surjective. On the other hand, as in the proof of Theorem 27, $h^0(X_n, \mathcal{O}_{X_n}(t)) = \binom{t+4}{4} - \binom{t+2}{4}$. Hence ρ''_t is bijective. It follows $\ker(\rho_t) = \ker(\rho'_t)$, i.e., every degree t hypersurface of \mathbb{P}^4 contains X_n if and only if it is a union of Q_n and a degree $t-2$ hypersurface of \mathbb{P}^4 .
- (B) Assume $t = q_0 + 1$. Proposition 14 and straightforward computations allow us to write the dimension of the vector space $L(t(q+2q_0+1)P_\infty)$ as

$$\dim_{\mathbb{F}} L(t(q+2q_0+1)P_\infty) = \binom{q_0+5}{4} - \binom{q_0+3}{4} - 4 = \binom{t+4}{4} - \binom{t+2}{4} - 4.$$

Since X_n is obtained by embedding C_n through the linear system $| (q+2q_0+1)P_\infty |$, we have also $h^0(X_n, \mathcal{O}_{X_n}(t)) = \binom{t+4}{4} - \binom{t+2}{4} - 4$. Since ρ'_t is surjective, $\dim_{\mathbb{F}} \ker(\rho'_t) = \binom{t+2}{4}$. As a consequence, we deduce the following inequality:

$$\begin{aligned} \dim_{\mathbb{F}} \ker(\rho_t) &\geq h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t-2)) - h^0(X_n, \mathcal{O}_{X_n}(t)) \\ &= \binom{t+4}{4} - \left[\binom{t+4}{4} - \binom{t+2}{4} - 4 \right] \\ &= \binom{t+2}{4} + 4 \\ &= \dim_{\mathbb{F}} \ker(\rho'_t) + 4. \end{aligned}$$

Since $\ker(\rho'_t) \subseteq \ker(\rho_t)$, there must exist at least four linearly independent hypersurfaces of \mathbb{P}^4 vanishing on X_n and not vanishing on Q_n , as claimed. \square

Example 29. By Proposition 18, a basis of the Riemann-Roch space $L((q+2q_0+1)P_\infty)$ is given by $\{1, x, y, v, w\}$. Taking homogeneous coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$ in \mathbb{P}^4 , we assume without loss of generality that X_n is the embedding of C_n defined by the following relations:

$$x_1/x_5 = x, \quad x_2/x_5 = y, \quad x_3/x_5 = v, \quad x_4/x_5 = w.$$

It is easily checked that the degree two hypersurface $Q_n \subseteq \mathbb{P}^4$ defined by the affine equation $x_2^2 = x_1x_3 + x_4$ contains X_n . By Corollary 28, Q_n is the unique degree two hypersurface of \mathbb{P}^4 containing X_n (its equation is defined up to a scalar multiplication). The equations of two linearly independent degree $q_0 + 1$ hypersurfaces of \mathbb{P}^4 and not containing Q_n appeared in the proof of Proposition 25, step (C.2):

$$x^{q_0+1} = v^{q_0} - y, \quad x^q y = w^{q_0} - v.$$

As pointed out in the Introduction, we find the same equations in [1], page 4.

Remark 30. Lemma 8 provides an explicit characterization of all the very ample linear systems of the form $|mP_\infty|$. We studied in details the case $m = q + 2q_0 + 1$, which provides the ‘smallest’ possible embedding of C_n . Other very ample linear systems can be considered, obtaining projective models of Suzuki curves in higher-dimensional projective spaces. We notice that the smallest $m > q + 2q_0 + 1$ such that $|mP_\infty|$ is very ample is $2q + 2q_0 + 1$. Moreover, $|(2q + 2q_0 + 1)P_\infty|$ embeds C_n into \mathbb{P}^9 . A systematic study of higher-degree embeddings seems to be difficult.

CONCLUSIONS

In this paper we construct projective smooth models of a plane Suzuki curve S_n through linear systems of the form $|mP_\infty|$, where P_∞ is the only singular point of any S_n . Computational results on the Weierstrass semigroup at P_∞ are applied in order to study in depth the smallest possible embedding X_n , in \mathbb{P}^4 , from a geometric point of view. In particular, the small-degree hypersurfaces of \mathbb{P}^4 containing X_n are characterized in Corollary 28, proving also that the same result cannot be extended to higher-degree hypersurfaces. Moreover, high-degree hypersurfaces of \mathbb{P}^4 containing X_n are explicitly counted. In order to derive such geometric results, here we solve some one-point Riemann-Roch problems in the range which is not trivially covered by the homonymous theorem, providing closed formulas.

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